Solution Key

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1. **Answer (D):** The machine worked for 2 hours and 40 minutes, or 160 minutes, to complete one third of the job, so the entire job will take $3 \cdot 160 = 480$ minutes, or 8 hours. Hence the doughnut machine will complete the job at 4:30 PM.

2. **Answer (A):** Let x be the side length of the square. Then the area of the square is x^2. The rectangle has sides of length $2x$ and $4x$, and hence area $8x^2$. The fraction of the rectangle's area inside the square is $\frac{x^2}{8x^2} = \frac{1}{8}$ or 12.5%.

3. **Answer (A):** The positive divisors of 6, other than 6, are 1, 2, and 3, so $<6> = 1 + 2 + 3 = 6$. As a consequence, we also have $<<<6>>> = 6$.

   Note: A positive integer whose divisors other than itself add up to that positive integer is called a perfect number. The two smallest perfect numbers are 6 and 28.

4. **Answer (C):** Note that $\frac{2}{3}$ of 10 bananas is $\frac{20}{3}$ bananas, which are worth as much as 8 oranges. So one banana is worth as much as $8 \cdot \frac{3}{20} = \frac{6}{5}$ oranges. Therefore $\frac{1}{2}$ of 5 bananas are worth as much as $\frac{5}{2} \cdot \frac{6}{5} = 3$ oranges.

5. **Answer (B):** Because each denominator except the first can be canceled with the previous numerator, the product is $\frac{2008}{4} = 502$.

6. **Answer (D):** Let x be the length of one segment, in kilometers. To complete the race, the triathlete takes

$$\frac{x}{3} + \frac{x}{20} + \frac{x}{10} = \frac{29}{60}x$$

   hours to cover the distance of $3x$ kilometers. The average speed is therefore

$$\frac{3x}{\frac{29}{60}x} \approx 6 \text{ kilometers per hour.}$$

7. **Answer (E):** First note that

$$\frac{\left(3^{2008}\right)^2 - \left(3^{2006}\right)^2}{\left(3^{2007}\right)^2 - \left(3^{2005}\right)^2} = \frac{9^{2008} - 9^{2006}}{9^{2007} - 9^{2005}}.$$

   Factoring 9^{2005} from each of the terms on the right side produces

$$\frac{9^{2008} - 9^{2006}}{9^{2007} - 9^{2005}} = \frac{9^{2005} \cdot 9^3 - 9^{2005} \cdot 9^1}{9^{2005} \cdot 9^2 - 9^{2005} \cdot 1} = \frac{9^{2005}}{9^{2005}} \cdot \frac{9^3 - 9}{9^2 - 1} = 9 \cdot \frac{9^2 - 1}{9^2 - 1} = 9.$$

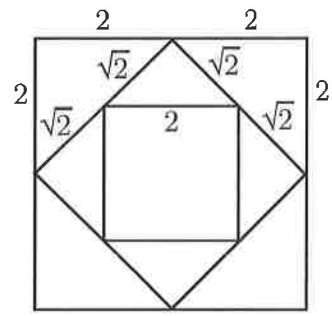
8. **Answer (A):** Let x denote the sticker price, in dollars. Heather pays $0.85x - 90$ dollars at store A and would have paid $0.75x$ dollars at store B. Thus the sticker price x satisfies $0.85x - 90 = 0.75x - 15$, so $x = 750$.

9. **Answer (B):** Because

$$\frac{2x}{3} - \frac{x}{6} = \frac{x}{2}$$

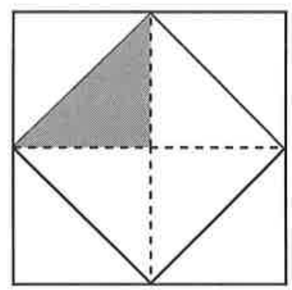
is an integer, x must be even. The case $x = 4$ shows that x is not necessarily a multiple of 3 and that none of the other statements must be true.

10. **Answer (E):** The sides of S_1 have length 4, so by the Pythagorean Theorem the sides of S_2 have length $\sqrt{2^2 + 2^2} = 2\sqrt{2}$. By similar reasoning the sides of S_3 have length $\sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$. Thus the area of S_3 is $2^2 = 4$.



OR

Connect the midpoints of the opposite sides of S_1. This cuts S_1 into 4 congruent squares as shown. Each side of S_2 cuts one of these squares into two congruent triangles, one inside S_2 and one outside.



Thus the area of S_2 is half that of S_1. By similar reasoning, the area of S_3 is half that of S_2, and one fourth that of S_1.

11. **Answer (D):** At the rate of 4 miles per hour, Steve can row a mile in 15 minutes. During that time $15 \cdot 10 = 150$ gallons of water will enter the boat. LeRoy must bail $150 - 30 = 120$ gallons of water during that time. So he must bail at the rate of at least $\frac{120}{15} = 8$ gallons per minute.

OR

Steve must row for 15 minutes to reach the shore, so the amount of water in the boat can increase by at most $\frac{30}{15} = 2$ gallons per minute. Therefore LeRoy must bail out at least $10 - 2 = 8$ gallons per minute.

12. **Answer (C):** Let b and g represent the number of blue and green marbles, respectively. Then $r = 1.25b$ and $g = 1.6r$. Thus the total number of red, blue, and green marbles is

$$r + b + g = r + \frac{r}{1.25} + 1.6r = r + 0.8r + 1.6r = 3.4r.$$

13. **Answer (D):** In one hour Doug can paint $\frac{1}{5}$ of the room, and Dave can paint $\frac{1}{7}$ of the room. Working together, they can paint $\frac{1}{5}+\frac{1}{7}$ of the room in one hour. It takes them t hours to do the job, but because they take an hour for lunch, they work for only $t-1$ hours. The fraction of the room that they paint in this time is

$$\left(\frac{1}{5}+\frac{1}{7}\right)(t-1),$$

which must be equal to 1. It may be checked that the solution, $t=\frac{47}{12}$, does not satisfy the equation in any of the other answer choices.

14. **Answer (D):** Let h and w be the height and width of the screen, respectively, in inches. By the Pythagorean Theorem, $h{:}w{:}27 = 3{:}4{:}5$, so

$$h=\frac{3}{5}\cdot 27 = 16.2 \quad \text{and} \quad w=\frac{4}{5}\cdot 27 = 21.6.$$

The height of the non-darkened portion of the screen is half the width, or 10.8 inches. Therefore the height of each darkened strip is

$$\frac{1}{2}(16.2-10.8) = 2.7 \quad \text{inches.}$$

OR

The screen has dimensions $4a \times 3a$ for some a. The portion of the screen not covered by the darkened strips has aspect ratio 2:1, so it has dimensions $4a \times 2a$. Thus the darkened strips each have height $\frac{a}{2}$. By the Pythagorean Theorem, the diagonal of the screen is $5a = 27$ inches. Hence the height of each darkened strip is $\frac{27}{10} = 2.7$ inches.

15. **Answer (D):** Suppose that Ian drove for t hours at an average speed of r miles per hour. Then he covered a distance of rt miles. The number of miles Han covered by driving 5 miles per hour faster for 1 additional hour is

$$(r+5)(t+1) = rt+5t+r+5.$$

Since Han drove 70 miles more than Ian,

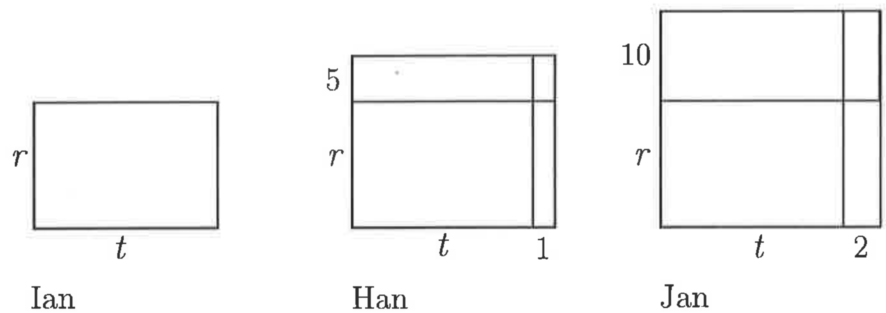
$$70 = (r+5)(t+1)-rt = 5t+r+5, \quad \text{so} \quad 5t+r = 65.$$

The number of miles Jan drove more than Ian is consequently

$$(r+10)(t+2)-rt = 10t+2r+20 = 2(5t+r)+20 = 2\cdot 65+20 = 150.$$

OR

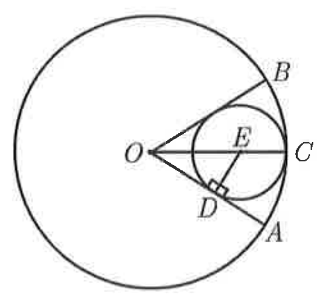
Represent the time traveled, average speed, and distance for Ian as length, width, and area, respectively, of a rectangle as shown. A similarly formed rectangle for Han would include an additional 1 unit of length and 5 units of width as compared to Ian's rectangle. Jan's rectangle would have an additional 2 units of length and 10 units of width as compared to Ian's rectangle.



Given that Han's distance exceeds that of Ian by 70 miles, and Jan's $10 \times t$ and $2 \times r$ rectangles are twice the size of Ian's $5 \times t$ and $1 \times r$ rectangles, respectively, it follows that Jan's distance exceeds that of Ian by

$$2(70 - 5) + 20 = 150 \text{ miles.}$$

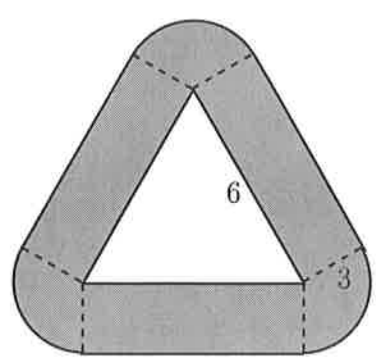
16. **Answer (B):** Let r and R be the radii of the smaller and larger circles, respectively. Let E be the center of the smaller circle, let $\overline{OC}$ be the radius of the larger circle that contains E, and let D be the point of tangency of the smaller circle to $\overline{OA}$. Then $OE = R - r$, and because $\triangle EDO$ is a $30-60-90°$ triangle, $OE = 2DE = 2r$. Thus $2r = R - r$, so $\frac{r}{R} = \frac{1}{3}$. The ratio of the areas is $(\frac{1}{3})^2 = \frac{1}{9}$.



17. **Answer (B):** The region consists of three rectangles with length 6 and width 3 together with three 120° sectors of circles with radius 3.

The combined area of the three 120° sectors is the same as the area of a circle with radius 3, so the area of the region is

$$3 \cdot 6 \cdot 3 + \pi \cdot 3^2 = 54 + 9\pi.$$



18. **Answer (B):** Let x be the length of the hypotenuse, and let y and z be the lengths of the legs. The given conditions imply that

$$y^2 + z^2 = x^2, \quad y + z = 32 - x, \quad \text{and} \quad yz = 40.$$

Thus

$$(32-x)^2 = (y+z)^2 = y^2 + z^2 + 2yz = x^2 + 80,$$

from which $1024 - 64x = 80$, and $x = \frac{59}{4}$.

Note: Solving the system of equations yields leg lengths of

$$\frac{1}{8}(69+\sqrt{2201}) \quad \text{and} \quad \frac{1}{8}(69-\sqrt{2201}),$$

so a triangle satisfying the given conditions does in fact exist.

19. **Answer (C):** Let P' and S' denote the positions of P and S, respectively, after the rotation about R, and let P'' denote the final position of P. In the rotation that moves P to position P', the point P rotates 90° on a circle with center R and radius $PR = \sqrt{2^2+6^2} = 2\sqrt{10}$. The length of the arc traced by P is $(1/4)\left(2\pi \cdot 2\sqrt{10}\right) = \pi\sqrt{10}$. Next, P' rotates to P'' through a 90° arc on a circle with center S' and radius $S'P' = 6$. The length of this arc is $\frac{1}{4}(2\pi \cdot 6) = 3\pi$. The total distance traveled by P is

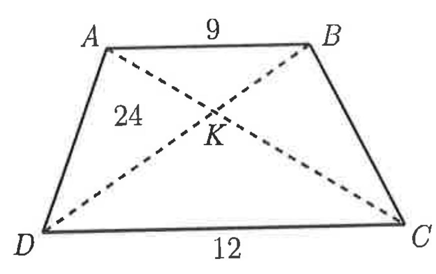
$$\pi\sqrt{10} + 3\pi = \left(3+\sqrt{10}\right)\pi.$$

20. **Answer (D):** Note that $\triangle ABK$ is similar to $\triangle CDK$. Because $\triangle AKD$ and $\triangle KCD$ have collinear bases and share a vertex D,

$$\frac{\text{Area}(\triangle KCD)}{\text{Area}(\triangle AKD)} = \frac{KC}{AK} = \frac{CD}{AB} = \frac{4}{3},$$

so $\triangle KCD$ has area 32.

By a similar argument, $\triangle KAB$ has area 18. Finally, $\triangle BKC$ has the same area as $\triangle AKD$ since they are in the same proportion to each of the other two triangles. The total area is $24 + 32 + 18 + 24 = 98$.



OR

Let h denote the height of the trapezoid. Then

$$24 + \text{Area}(\triangle AKB) = \frac{9h}{2}.$$

Because $\triangle CKD$ is similar to $\triangle AKB$ with similarity ratio $\frac{12}{9} = \frac{4}{3}$,

$$\text{Area}(\triangle CKD) = \frac{16}{9}\text{Area}(\triangle AKB), \quad \text{so} \quad 24 + \frac{16}{9}\text{Area}(\triangle AKB) = \frac{12h}{2}.$$

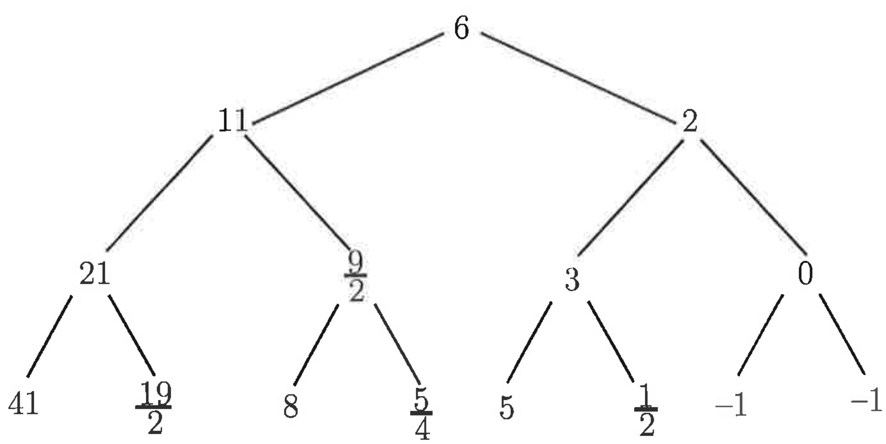
Solving the two equations simultaneously yields $h = \frac{28}{3}$. This implies that the area of the trapezoid is

$$\frac{1}{2} \cdot \frac{28}{3}(9+12) = 98.$$

21. **Answer (A):** All sides of $ABCD$ are of equal length, so $ABCD$ is a rhombus. Its diagonals have lengths $AC = \sqrt{3}$ and $BD = \sqrt{2}$, so its area is

$$\frac{1}{2}\sqrt{3} \cdot \sqrt{2} = \frac{\sqrt{6}}{2}.$$

22. **Answer (D):** The tree diagram below gives all possible sequences of four terms. In the diagram, each left branch from a number corresponds to a head, and each right branch to a tail.

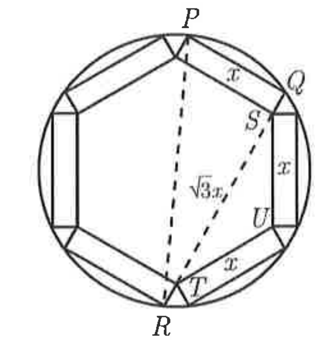


Because the coin is fair, each of the eight possible outcomes in the bottom row of the diagram is equally likely. Five of those numbers are integers, so the required probability is $\frac{5}{8}$.

23. **Answer (B):** Let the two subsets be A and B. There are $\binom{5}{2} = 10$ ways to choose the two elements common to A and B. There are then $2^3 = 8$ ways to assign the remaining three elements to A or B, so there are 80 ordered pairs (A, B) that meet the required conditions. However, the ordered pairs (A, B) and (B, A) represent the same pair $\{A, B\}$ of subsets, so the conditions can be met in $\frac{80}{2} = 40$ ways.

24. **Answer (D):** The units digit of 2^n is 2, 4, 8, and 6 for $n = 1$, 2, 3, and 4, respectively. For $n > 4$, the units digit of 2^n is equal to that of 2^{n-4}. Thus for every positive integer j the units digit of 2^{4j} is 6, and hence 2^{2008} has a units digit of 6. The units digit of 2008^2 is 4. Therefore the units digit of k is 0, so the units digit of k^2 is also 0. Because 2008 is even, both 2008^2 and 2^{2008} are multiples of 4. Therefore k is a multiple of 4, so the units digit of 2^k is 6, and the units digit of $k^2 + 2^k$ is also 6.

25. **Answer (C):** Select one of the mats. Let P and Q be the two corners of the mat that are on the edge of the table, and let R be the point on the edge of the table that is diametrically opposite P as shown. Then R is also a corner of a mat and $\triangle PQR$ is a right triangle with hypotenuse $PR = 8$. Let S be the inner corner of the chosen mat that lies on $\overline{QR}$, T the analogous point on the mat with corner R, and U the corner common to the other mat with corner S and the other mat with corner T. Then $\triangle STU$ is an isosceles triangle with two sides of length x and vertex angle 120°. It follows that $ST = \sqrt{3}x$, so $QR = QS+ST+TR = \sqrt{3}x+2$. Apply the Pythagorean Theorem to $\triangle PQR$ to obtain $\left(\sqrt{3}x + 2\right)^2 + x^2 = 8^2$, from which $x^2 + \sqrt{3}x - 15 = 0$. Solve for x and ignore the negative root to obtain



$$x = \frac{3\sqrt{7} - \sqrt{3}}{2}.$$